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Synchronization of chaotic systems by continuous control

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(Received 13 October 1995)

We investigate the synchronization of two chaotic systems through continuous feedback control. The dependence of the synchronization efficiency on the perturbation weight is studied, as well as the influence of noise.

PACS number(s): 05.45.+b

Due to the sensitive dependence of chaotic dynamical systems on the initial conditions, control of chaos may appear a remote possibility. Moreover, in practical applications, system imperfect identification and the ubiquitous presence of internal and external noise seem to make the task even more hopeless. Nonetheless, it has been established that synchronization of chaotic systems is possible [1,2] and has practical potential applications [3]. Although the dynamics of a chaotic system can be modified by time-dependent signals (nonfeedback control [4]), it is generally accepted that feedback control [5,6] offers more advantages. In their pioneering work Ott, Grebogi, and York [5] showed that by applying small temporal-parameter perturbations it is possible to stabilize unstable periodic orbits embedded in the chaotic attractor. This method was later extended [7] to achieve synchronization of a chaotic trajectory of one system about a chaotic trajectory of another system. Though the application of the Ott-Grebogi-York method requires a permanent analysis of the state of the system, the changes of the parameter are discrete in time since the method deals with the Poincaré map. It follows that the presence of noise may lead to occasional bursts of the system into regions far from the desidered orbit. To overcome such limitations Pyragas [6] suggested an alternative approach to chaos control based on stabilization of unstable periodic orbits through a small time continuous perturbation (either by combined feedback with the use of an external oscillator or by delayed self-controlling feedback). The continuous control does not require any computer analysis of the system and can be particularly convenient for experimental applications. Recently Kapitaniak [8] suggested the possibility of applying this method to achieve synchronization of two chaotic systems and reported some preliminary results. Here we present a more detailed study of synchronization through a continuous chaos control. We analyze the dependence of the synchronization efficiency on the perturbation strength and investigate the influence of noise as well as that of the choice of the variable used to implement the method.

The continuous feedback control method can be summarized as follows. We consider two identical chaotic dynamic systems that can be simulated by the equations

$$\dot{x} = f(x), \quad \dot{y} = f(y), \tag{1}$$

with x,$y \in \mathbb{R}^n$, called A and B, respectively. Let us suppose that some state variable of both systems can be measured, for example, $x_i(t)$ from system A and $y_i(t)$ from system B ($i = 1, ..., n$). To achieve synchronization, we can use the differerence

$$F(t) = K[y_i(t) - x_i(t)] \tag{2}$$

as a negative feedback introduced into one of the chaotic systems (A in our case). Here K is a positive quantity representing an experimentally adjustable weight of the perturbation. The control signal (2) forces the solution of system A over that of system B, so that synchronization eventually follows. Since in this regime $x_i(t) = y_i(t)$, $F(t)$ becomes zero and the two systems are practically uncoupled, thus obeying the same dynamics as in the absence of the perturbation.

We investigate the above outlined synchronization procedure choosing as a model system the Duffing oscillator

$$\ddot{x} + a\dot{x} + x^3 = B \cos t. \tag{3}$$

The parameters a and B were chosen so as to correspond to a chaotic behavior ($a = 0.1, B = 10$). Consequently, the trajectories of two such Duffing oscillators with slightly different initial conditions diverge exponentially from each other. Synchronization can be achieved by coupling them through a feedback control $F(t)$ to obtain the system [8]

$$\begin{aligned} \ddot{x} + a\dot{x} + x^3 &= B \cos t + F(t), \\ \ddot{y} + a\dot{y} + y^3 &= B \cos t. \end{aligned} \tag{4}$$

We exploited a numerical solution of system (4) us-

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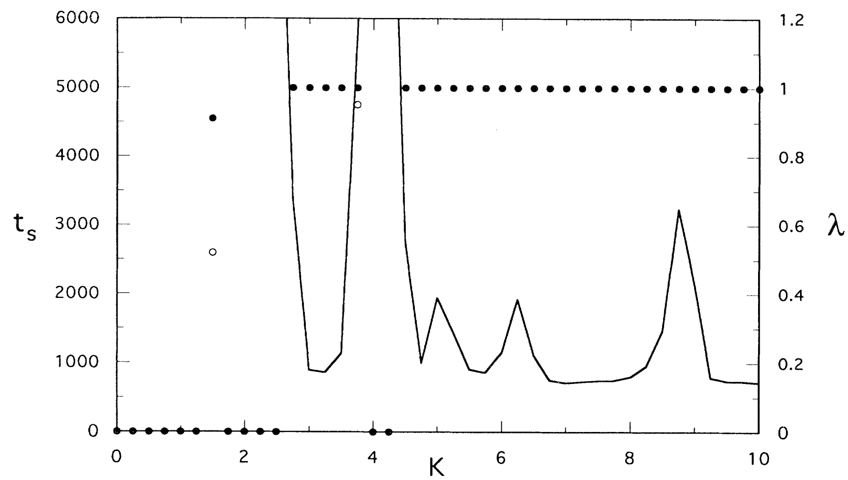

FIG. 1. Plot of t_s (solid line) and λ (discrete markers) versus coupling stiffness K, for $F(t) = K(y - x)$. Results were averaged over 100 independent runs with a run length of 10^6 iterations (circles) and 2×10^6 iterations (dots). Where only dots are shown, these coincide with circles. The values of t_s are independent of the run length, at least on the scale adopted.

ing the fourth-order Runge-Kutta method. Calculations were carried out in double precision [9] with a time step $dt = 0.01$ (test runs were also performed with a time step of 0.001). The synchronization efficiency, for a given perturbation, is investigated in terms of the synchronization time t_s defined as the time taken in order that the distance $d(t)$ in the phase space between the orbits of the two coupled oscillators is of the order of the precision of the computers used, i.e., $d(t_s) < \epsilon$, where $d(t) = \{[x(t) - y(t)]^2 + [\dot{x}(t) - \dot{y}(t)]^2\}^{1/2}$ and $\epsilon = 10^{-14}$.

Let us first take $F(t) = K(y - x)$, which amounts to choosing the oscillator position as the output variable. Figure 1 shows the corresponding synchronization time as a function of the perturbation weight. For each value of K the results were averaged over $N = 100$ independent runs with randomly chosen initial conditions. Two sets of calculations were performed, with run lengths of 10^6 and $2\text{x}10^6$ iterations, respectively. In the figure we show also the fraction $\lambda = N_s/N$, where N_s is the number of runs for which synchronization is attained, with the required precision, within the maximum run length ($T_{\max}$) allowed. When for a given K synchronization is not attained for any of the runs performed, i.e., when $\lambda = 0$, it follows that t_s is *at least* greater than $T_{\max}$. The smallest value of K for which λ is different from zero is $K = 1.5$ [10]. We observe that λ is smaller than one but tends to this value as the run length increases. Thus synchronization can be achieved, though with a low efficiency: the corresponding t_s [11] is, in fact, quite large (out of the scale adopted in the figure). Synchronization is more efficiently attained for $K > K_s$, where $K_s \approx 2.5$, with the exception of a restricted interval around $K = 4$ (on this we will comment later). We note that, except for $K = 1.5$, both the values of t_s and those of λ are substantially stable for the two sets of calculations performed.

To support the preceding conclusions we investigated the stability of system (4) studying the dependence on K of the spectrum of the Lyapunov exponents through the method of Wolf *et al.* [12]. In order to ensure that the estimate of the Lyapunov exponents (L) is independent of the run length, for each value of K we performed two runs of 10^6 and 10^7 iterations, respectively, obtaining essentially identical results. We focus our attention on the two L's which at $K = 0$ are positive, their value coinciding with that of the largest Lyapunov exponent of a single Duffing oscillator in the chaotic regime considered (see Fig. 2). As K increases, one of the two L's remains essentially unaltered, whereas the other one gradually decreases. This behavior signals that while subsystem B is

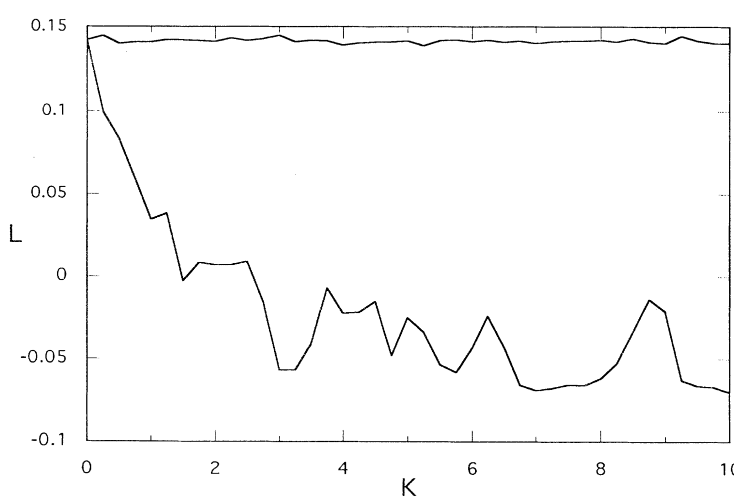

FIG. 2. The two largest Lyapunov exponents L of system (4), with $F(t) = K(y - x)$, versus coupling stiffness K. Results were obtained performing one single run of 10^7 iterations for each value of K.

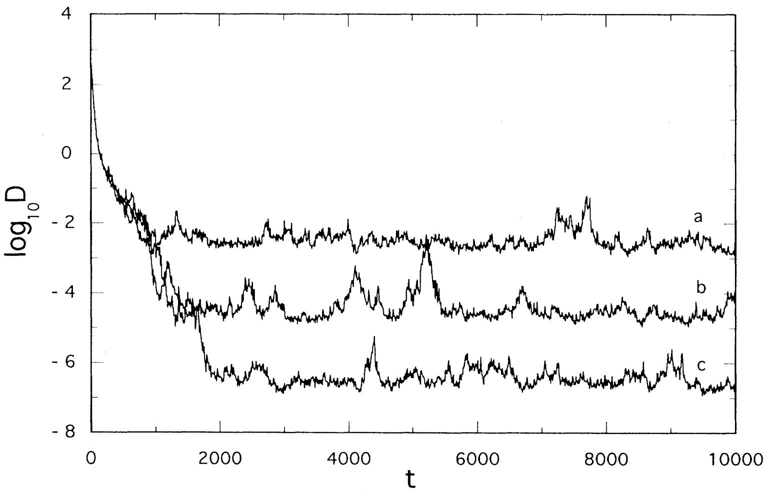

FIG. 3. Decimal logarithm of the mean square distance D between the orbits of subsystems A and B versus time, for $F(t) = K(y-x)$ with $K = 5$ and for different noise amplitudes W: $W = 10^{-2}$ (curve a), $W = 10^{-4}$ (curve b), and $W = 10^{-6}$ (curve c). Results were averaged over 100 independent realizations.

unaffected by the perturbation, subsystem A becomes less and less unstable. Note that for all positive values of K the perturbation causes a decrease of the Lyapunov exponent corresponding to subsystem A (called in the following L_A) with respect to the nonperturbed case, but an inversion in its sign occurs *only* for a sufficiently large perturbation weight. In fact, as K increases, L_A becomes negative first for $K \approx 1.5$ and then for $K > K_s$. Comparing Figs. 1 and 2 one observes that stability is a necessary condition in order that synchronization can be attained. Moreover, if L_A is negative *and* synchronization is attained, the greater the absolute value of L_A, the smaller t_s is. Stability, however, is not a sufficient condition either: in the region around $K = 4$, where $\lambda = 0$, in spite of a negative L_A, no synchronization is attained. Indeed, a negative value of L_A signals that orbits of subsystem A starting from different initial conditions coalesce into the same final orbit. Usually, at least in the range of values of K explored, this coincides with the orbit followed by subsystem B and thus synchronization ensues, but this is not always the case. In fact, in the above-mentioned region the final orbits of the two subsystems are different, though we observed that the perturbation introduces a dependence of the final orbit of subsystem A on the initial conditions of subsystem B.

Confronting the preceding results with those obtained by Kapitaniak [8] for the same system, it appears evident that synchronization is signaled in that paper at much smaller values of K. Moreover, at difference with our results (and also with those of Pyragas [6]), no threshold effect seems to be expected since the estimated synchronization time is roughly proportional to some inverse power of K. The explanation for the discrepancies is probably to be found in the criterion, much less demanding than ours, adopted by Kapitaniak [8] to calculate the synchronization time. In fact, the relatively large value of ϵ (10^{-4}) chosen in Ref. [7] may not allow a correct estimate of the synchronization time. Indeed it may happen that the orbits stay relatively close for a large number of iterations (thus *temporarily* satisfying a loose synchronization criterion) and then separate again.

To investigate the effectiveness of the continuous control synchronization method in noisy situations, we added a white noise to subsystem B and studied how the mean square distance D between the orbits of the two subsystems evolves with time for different noise amplitudes W and for a number of values of K (we report in Fig. 3 the results corresponding to $K = 5$). It appears evident that, for any given noise amplitude, D decreases with respect to its initial value, but is always greater than

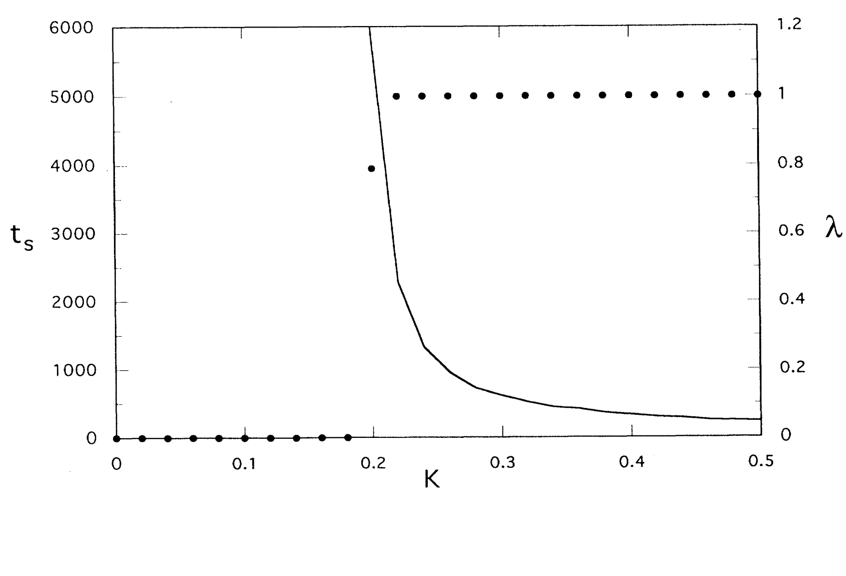

FIG. 4. Plot of t_s (solid line) and λ (dots) versus coupling stiffness K, for $F(t) = K(\dot{y} - \dot{x})$. Results were averaged over 100 independent runs of 10^6 iterations.

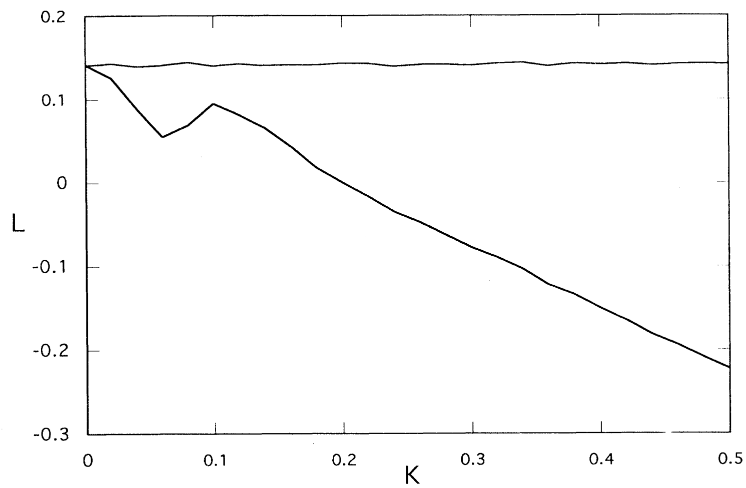

FIG. 5. The two largest Lyapunov exponents of system (4), with $F(t) = K(\dot{y} - \dot{x})$, versus coupling stiffness K. Results were obtained performing one single run of 10^7 iterations for each value of K.

a quantity of order 10^{-1} W. Thus the presence of noise does not destroy the synchronizing effect of the feedback control but limits the "degree" of synchronization attainable. Indeed, the synchronization criterion hereby adopted cannot be satisfied (unless 10^{-1} W $\leq \epsilon$) and consequently, the synchronization time tends to an infinite value. A rather different conclusion was reached by Kapitaniak [8]: according to his findings synchronization is still possible in the presence of noise and t_s remains essentially unaltered. However, this result, apparently in contrast with our calculations, was made possible *only* by his particular choice of parameters, the noise amplitude being just one order of magnitude greater than the value of ϵ employed in the synchronization criterion.

As outlined before, in principle it is possible to use any accessible state variable in order to implement the continuous feedback control method. Referring to the model system investigated in this Brief Report, we can choose as output variable the oscillator velocity instead of its position. This amounts to taking $F(t) = K(\dot{y} - \dot{x})$. In Figs. 4 and 5 the corresponding synchronization time and the two largest Lyapunov exponents of system (4) are shown as a function of K. A comparison with Figs. 1 and 2 makes evident an overall much smoother dependence on the perturbation weight and, which is most relevant, a synchronization threshold about one order of magnitude smaller than in the first case. Thus the synchronization efficiency may depend sensibly on the choice of the output variable. This conclusion is confirmed by some preliminary results concerning the Lorenz system [13]. As a general rule, the optimal choice of the output variable should realize a compromise between accessibility of the variable and synchronization efficiency.

In conclusion, we presented an investigation of a method for achieving synchronization of two chaotic systems, based on a continuous feedback control [6]. At difference with a previous study [8], our results point out that synchronization can be achieved only if the perturbation weight is greater than a threshold value. Moreover, we showed that the presence of noise limits the "accuracy" of synchronization, since the long-time distance between the orbits of the two systems is roughly of the order of the noise amplitude.

This work was supported by the Ministero dell'Università e della Ricerca Scientifica e Tecnologica and by the Consorzio Interuniversitario Nazionale per la Fisica della Materia.

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[9] On the RISC processor based computers used, this corresponds to a precision of about 10^{-15}.
[10] We adopted a grid of K values with step $dK = 0.25$.
[11] Since synchronization is not attained within $T_{\max}$ for all the runs performed, i.e., $\lambda < 1$, we calculated a lower estimate of t_s.
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